For the rest of the course we study plane curves. We start with their local properties in this chapter before looking at proj. curves in the next one.

4.1 Affine plane curves:

Informally an affine plane curve is simply the 0-set $V(F)$ of a non-constant polynomial $F \in k[X,Y]$. However we want to remember not only $V(F)$, but rather F itself.

Def: · $F, G \in k[X,Y]$ are <u>equivalent</u> if $F = \lambda \cdot G$ for some $\lambda \in k^\times$

· An <u>affine plane curve</u> is an equivalence class of non-const. polynomials under this relation.

· If $F = \prod_i F_i^{e_i}$ with F_i the irred. factors of F, we call F_i the <u>components</u> of F and e_i their <u>multiplicities</u>.

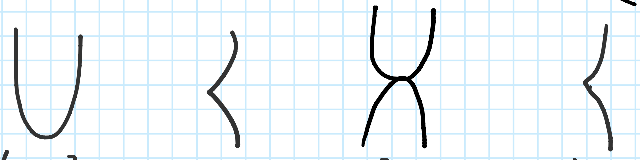
Rmk: We can recover the F_i up to equivalence from $V(F)$, but not the e_i.

· The <u>degree</u> of C is the degree of a defining $F \in k[X,Y]$

· A <u>line</u> is a curve of degree 1, i.e. of the form $F(X,Y) = aX + bY + c$.

· If F is irreducible, $V(F)$ is a variety and we write $\Gamma(F), k(F), \mathcal{O}_P(F)$ instead of $\Gamma(V(F)), k(V(F)), \mathcal{O}_P(V(F))$.
$= \mathcal{O}(V(F))$

<u>Exples:</u>



$Y - X^2$ $\quad$ $Y^2 - X^3$ $\quad$ $Y^2 - X^4$ $\quad$ $Y^2 - X^5$

Rmk: We will often abuse notation and write F both for a polynomial in $k[X,Y]$ and the (affine plane) curve it defines.

Def: Let F be a curve, $P = (a,b) \in F$ a point. Then P is called

Def: Let F be a curve, $P=(a,b)\in F$ a point. Then P is called a simple point if either

$$F_X(P):=\frac{\partial F}{\partial X}(P)\neq 0 \quad \text{or} \quad F_Y(P):=\frac{\partial F}{\partial Y}(P)\neq 0.$$

I.e. if the Jacobian of $F:\mathbb{A}^2\to\mathbb{A}^1$ has full rank at P.
In this case the line $L(X,Y)=F_X(P)(X-a)+F_Y(P)(Y-b)$ is the tangent line to F at P.

- A point that's not simple is called singular or multiple
- A curve with only simple points is called non-singular

Exple: $F(X,Y)=Y-X^2$ $\quad F_X=-2X$, $F_Y=1\neq 0\ \forall P\in F \to F$ is non-sing.

- $F(X,Y)=Y^2-X^3$ $\quad P\in F$ is singular $\Leftrightarrow P=(0,0)$.

We'll usually arrange things so that $P=(0,0)$ is a singular point of F, so let's look at the above def's in this case:

Write $F=F_m+F_{m+1}+\dots+F_n$, with $m\leq n$, F_i a form of degree i and $F_m\neq 0$

Def: The integer $m=:m_P(F)$ is called multiplicity of F at $P=(0,0)$

We have: $P=(0,0)\in F \Leftrightarrow m\geq 1$

- $P=(0,0)$ simple $\Leftrightarrow m=1$ and in this case F_1 is the tangent line to F at P.

If $m=2$, P is a double point
$\quad m=3$ " tripple point etc...

Since F_m is a form in two variables we can write

$F_m=\prod L_i^{r_i}$ where L_i are distinct lines through the origin. To see this, notice that the dehomogenization $(F_m)_*:=F_m(X,1)$ factors into linear terms since k is alg closed.

Def: The L_i are called tangent lines to F at P.

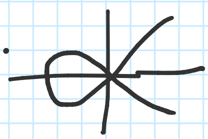
Def: • The L_i are called <u>tangent lines</u> to F at P.

• L_i is a <u>simple</u> (resp. double, tripple..) tangent if $r_i = 1$ $(2, 3 \dots)$

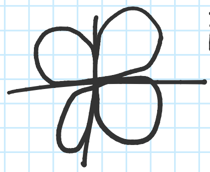
• P is an <u>ordinary multiple point</u> of F if F has m distinct tangents at P.

• An ordinary double point is called a <u>node</u>. It's in many ways the simplest example of a singular pt.

<u>Exple:</u> $F = Y^2 - X^3 - X^2 = F_2 + F_3 \rightsquigarrow$



$F_2 = Y^2 - X^2 = (\underset{L_1}{\underbrace{Y - X}})(\underset{L_2}{\underbrace{Y + X}}) \rightsquigarrow F$ has a node at $(0,0)$



$F = (X^2 + Y^2)^3 - 4X^2Y^2 = F_4 + \overset{0}{F_5} + F_6 \rightsquigarrow m_{(0,0)}(F) = 4$

$F_4 = -4\underset{L_1^2}{X^2}\underset{L_2^2}{Y^2} \rightsquigarrow F$ has two tangents of mult. 2

$\rightsquigarrow (0,0)$ is not an ordinary multiple point.

If $F = \prod F_i^{e_i}$, then $m_P(F) = \sum e_i \cdot m_P(F_i)$ $(P = (0,0))$, and if L is a tangent to F_i w/ mult r_i, then L is tangent to F w/ mult. $\sum_i e_i \cdot r_i$. This follows from the fact, that the lowest degree form of F is the product of the ones of the $F_i^{e_i}$'s.

Finally for $P = (a,b) \neq 0$ let $T(x,y) = (x+a, y+b)$ be the translation $0 \mapsto (a,b)$ and set $F^T(X,Y) = F(X+a, Y+b)$. We define $m_P(F) := m_{(0,0)}(F^T)$ and similarly for tangent lines, their multiplicities etc...

<u>Thm 4.1:</u> Let P be a point on an irred. curve F and $\mathfrak{m}_P(F) \subset \mathcal{O}_P(F)$ the corresp. maximal ideal. Then for all n sufficiently large (in fact $n \geq m_P(F)$) we have

$$m_P(F) = \dim_k\left(\mathfrak{m}_P(F)^n / \mathfrak{m}_P(F)^{n+1}\right)$$

Expl: $F(X,Y) = X$, $P = (0,0)$ $\rightsquigarrow$ $m_P(F) = (Y,X) \subset \left(k[X,Y]/X\right)_{(X,Y)}$

$\rightsquigarrow$ $\overset{\shortparallel}{(Y)} \subset k[Y]_{(Y)}$.

$\rightsquigarrow m_P(F)^n = (Y^n) \supset (Y^{n+1})$

and $(Y^n)/(Y^{n+1})$ is generated as a k-V.sp. by Y^n.

$\rightsquigarrow m_P(F) = 1$ as expected.

First we need a lemma (compare with Ex 2.5)

Lemma 4.2: Let $I \subset k[X_1, .., X_n]$ be an ideal s.t. $V(I) = \{P_1, .., P_N\}$ is a finite set of points and let $\mathcal{O}_i = \mathcal{O}_{P_i}(\mathbb{A}^n)$. Then there is a natural isomorphism

$$\varphi : k[X_1, .., X_n]/I \xrightarrow{\sim} \prod_{i=1}^{N} \mathcal{O}_i/I\mathcal{O}_i$$

pf: Let $m_i = m_{P_i}$ be the maximal ideal defining P_i and abbreviate $R = k[X_1..,X_n]/I$, $R_i = \mathcal{O}_i/I\mathcal{O}_i$.

For each i, I is in the kernel of

$$k[X_1,..,X_n] \hookrightarrow \mathcal{O}_i \twoheadrightarrow \mathcal{O}_i/I \cdot \mathcal{O}_i = R_i$$

$\rightsquigarrow \varphi_i : R \to R_i$ and we define φ as the product of the φ_i.

To show that φ is an iso., we will construct "idempotents":

i.e. $e_1, \dots, e_N \in R$ s.t. 1) $e_i^2 = e_i$, $e_i \cdot e_j = 0$ and $\sum_{i=1}^{N} e_i = 1$

2) $e_i(P_i) = 1$ and $\forall G \in k[X_1,.., X_n]$ w/ $G(P_i) \neq 0$ $\exists t \in R$ s.t. $t \cdot g = e_i$.

$\Uparrow$ G is a unit in $\mathcal{O}_i$.

Assume we have this. Then

φ is injective: $f \in \ker(\varphi_i) \iff \exists G$ w/ $G(P_i) \neq 0$ and $G \cdot F \in I$.

$\rightsquigarrow f \in \ker(\varphi) \rightsquigarrow \forall i$ $\exists G_i$ w/ $G_i(P_i) \neq 0$ and $G_i \cdot F \in I$.

$\rightsquigarrow f \overset{1)}{=} \sum e_i \cdot f \overset{2)}{=} \sum t_i \cdot g_i \cdot f = 0$.

$\leadsto f \overset{1)}{=} \sum e_i \cdot f \overset{2)}{=} \sum t_i \cdot \underbrace{g_i \cdot f}_{=0} = 0.$

<u>φ is surj</u>: Let $z = \left(\frac{a_1}{g_1}, \dots, \frac{a_N}{g_N}\right) \in \prod R_i$ w/ $a_i, g_i \in R$ and $g_i(P_i) \neq 0$.

Let $t_i \in R$ be s.t. $t_i \cdot g_i = e_i$.

Since $e_i(P_i) = 1$, $\varphi_i(e_i) \in R_i$ is a unit.

$\leadsto \varphi_i(e_j) = \varphi_i(\underbrace{e_j \cdot e_i}_{=0}) \cdot \varphi_i(e_i)^{-1} = 0 \leadsto \varphi_i(e_i) = \varphi_i(\sum e_j) = 1.$

$\leadsto \varphi_i(t_i \cdot g_i) = 1 \leadsto \frac{a_i}{g_i} = \varphi_i(a_i \cdot t_i)$

and finally $\varphi_i\left(\sum_{j=1}^{N} a_j t_j e_j\right) = \frac{a_i}{g_i} \quad \forall i$ ✓

It remains to construct $e_1, \dots, e_N$:

By the NSS $\mathrm{Rad}(I) = I(P_1, \dots P_N) = \bigcap_{i=1}^{N} m_i$

$\leadsto \exists d$ s.t. $(\bigcap m_i)^d \subset I$

Choose $F_1, \dots, F_N \in k[X_1, \dots, X_n]$ s.t. $F_i(P_j) = \partial_{i,j}$

and set $E_i = 1 - (1 - F_i^d)^d$. Then the residues e_i of F_i in R satisfy 1) and 2).

<u>1)</u> We have $E_i = F_i^d \cdot D_i \leadsto E_i \in m_j^d \quad \forall j \neq i$

$\leadsto i \neq j \quad E_i \cdot E_j \in \bigcap_{k=1}^{N} m_k^d = (\bigcap m_k)^d \subset I, \quad e_i \cdot e_j = 0.$

(Exercise)

$$1 - \sum_{i=1}^{N} E_i = \underbrace{(1 - E_j)}_{= (1 - F_j^d)^d \in m_j^d} - \underbrace{\sum_{i \neq j} E_i}_{\in m_j^d} \in \bigcap m_k^d \subset I \leadsto \sum e_i = 1$$

$$E_i - E_i^2 = E_i (1 - F_i^d)^d \in \bigcap_{j \neq i} m_j^d \cdot m_i^d \subset I.$$

2) Clearly $e_i(P_i) = E_i(P_i) = 1$. Now let $G \in k[X_1, \dots, X_n]$

3) Clearly $e_i(P_i) = E_i(P_i) = 1$. Now let $G \in K[X_1, \dots, X_n]$ w/ $G(P_i) \neq 0$, say $G(P_i) = 1$. Then $1 - G \in m_i$

$\rightsquigarrow (1-G)^d E_i \in m_i^d \cdot \bigcap_{j \neq i} m_j^d \in I \rightsquigarrow e_i (1-g)^d = 0$ in R

$\rightsquigarrow e_i = g(\quad)$ Lemma $\square$

<u>Pf of thm 4.1:</u> Let's write $\mathcal{O}$, m for $\mathcal{O}_P(F)$ and $m_P(F)$.

We have an exact sequence of $\mathcal{O}$-modules (or K-vect.)

$$0 \to m^n/m^{n+1} \to \mathcal{O}/m^{n+1} \to \mathcal{O}/m^n \to 0$$

$\rightsquigarrow \dim_K(m^n/m^{n+1}) = \dim_K(\mathcal{O}/m^{n+1}) - \dim_K(\mathcal{O}/m^n)$

$\rightsquigarrow$ The thm follows if $\dim_K(\mathcal{O}/m^n) = n \cdot m_P(F) + s$ for some $s \in \mathbb{Z}$ and all $n \geq m_P(F)$.

W.l.o.g. $P = (0,0)$ and hence $m^n = I^n \mathcal{O}$ w/ $I = (X,Y) \subset K[X,Y]$.

Then $\mathcal{O}/m^n = \mathcal{O}/I^n\mathcal{O} \underset{\nearrow}{=} \mathcal{O}_P(\mathbb{A}^2)/(I^n, F) \underset{\uparrow}{\cong} K[X,Y]/(I^n, F)$

loc. commutes w/ taking quotients $\quad$ Lemma

$\rightsquigarrow \dim_K(\mathcal{O}/m^n) = \dim_K(K[X,Y]/(I^n, F))$.

Let $m_P(F) = m$. Then $\forall G \in I^{n-m}$ we have $F \cdot G \in I^n$

$\rightsquigarrow$ We get an exact sequence

$$0 \to K[X,Y]/I^{n-m} \xrightarrow{\psi} K[X,Y]/I^n \xrightarrow{\varphi} K[X,Y]/(I^n, F) \to 0$$
$$\overline{G} \longmapsto \overline{GF}.$$

Since $\dim_K K[X,Y]/I^n = \frac{n(n+1)}{2}$ (Ex. 8.1) we have

$\dim_K(K[X,Y]/(I^n, F)) = n \cdot m - \frac{m(m-1)}{2}$ $\square$

4.2. Curves and discrete valuation rings

The local rings at a simple point of a plane curve have a very special structure.

<u>Proposition 4.3:</u> Let R be a domain that is not a field. Then the following are equivalent:

1) R is Noetherian, local and the max. ideal is principal

2) There exists an irreducible $t \in R$ s.t. every $z \in R \setminus \{0\}$ can be written uniquely as $z = u \cdot t^n$ where $u \in R$ is a unit and $n \in \mathbb{Z}_{\geq 0}$.

A ring that satisfies these properties is called a <u>discrete valuation ring</u> or DVR.

<u>pf:</u> <u>1=>2:</u> Let $m \subset R$ be the max. ideal and $t \in m$ a generator. Since $(t) = m$ is maximal, t is irred.

Assume $u \cdot t^n = v \cdot t^m$ w/ $n \geq m$. Then

$$u \cdot t^{n-m} = v \in R^\times = R \setminus m$$

$\rightsquigarrow n = m \rightsquigarrow u = v. \rightsquigarrow$ uniqueness ✓.

Assume $\exists z \in R \setminus \{0\}$ that is not of the form $u \cdot t^n$.

$\rightsquigarrow z$ not a unit, so $z \in m = (t)$

$\rightsquigarrow \exists z_1 \in R$ s.t. $z = t \cdot z_1$. If z_1 is a unit we're done, otherwise we get $z_2, \dots$ w/ $z_i = t \cdot z_{i+1}$.

Since R is Noetherian, the chain of ideals

$(z) \subset (z_1) \subset (z_2) \subset \dots$ stabilizes $\Rightarrow \exists n$ s.t. $(z_n) = (z_{n+1})$

$\rightsquigarrow \exists v \in R$ s.t. $z_{n+1} = v \cdot z_n = v \cdot t \cdot z_{n+1} \rightsquigarrow v \cdot t = 1$ ↯ $t \in m$.

2) ⟹ 1): Clearly $m := (t)$ is the set of non-units.

$\rightsquigarrow$ R is local w/ max ideal m.

To show that R is Noetherian it's enough to show that every ideal in R is fin. generated.

Let $I \subset R$ an ideal, then $I \subset m$.

Let $n = \min\{n \mid \exists u \in R^\times \text{ s.t. } u \cdot t^n \in I\}$, then $t^n \in I$

But any $z \in I$ is of the form $u \cdot t^m$ w/ $m \geq n$

$\rightsquigarrow (t^n) = I$, i.e. I is finitely generated. □

Rmk: The proof shows in part that any ideal in R is of the form $(t^n) = m^n$ for some n.

Any DVR is a PID.

Expls: $\cdot\ k[x]_{(x)} = \{\frac{f}{g} \mid g(0) \neq 0\}$.

$\cdot\ k[[t]]$ w/ max ideal (t)

$\cdot\ \mathbb{Z}_{(p)} = \{\frac{r}{s} \mid r, s \in \mathbb{Z}, p \nmid s\}$

Non-example: $k[X,Y]_{(X,Y)}$

Def: A generator t of the maximal ideal in a DVR is called a <u>uniformizing parameter</u> or <u>uniformizer</u>.

Any two differ by a unit in R.

If $K = Q(R)$ denotes the quotient field, then any $z \in K^\times$ can be written uniquely as $z = u \cdot t^n$ w/ $u \in R^\times$ and $n \in \mathbb{Z}$. The integer n is called the <u>order of z</u>, and is denoted by $\mathrm{ord}(z)$.

Corollary 4.4: Let F be an irred. plane curve and $P \in F$.

Then P is simple $\Leftrightarrow \mathcal{O}_P(F)$ is a DVR.

In this case, if $L = aX + bY + c$ is any line through P that

In this case, if $L = aX + bY + c$ is any line through P that is not tangent to F at P, then the image of L in $\mathcal{O}_P(F)$ is a uniformizer for $\mathcal{O}_P(F)$.

<u>Def</u>: If $p \in F$ is simple, F irred., we write
ord_P^F for <u>the order</u> on $k(F)$ induced by $k(F) = Q(\mathcal{O}_P(F))$

We have $\operatorname{ord}_P^F(g) = \dim_k\left(\mathcal{O}_P(F)/(g)\right)$ for any $g \in \mathcal{O}_P(F)$.

$\ulcorner d = \operatorname{ord}_P^F(g) \rightsquigarrow (g) = (t^d)$ and $\dim_k\left(\mathcal{O}_P(F)/(t^d)\right) = \underbrace{\dim_k(\mathcal{O}_P(F)/t) + \dim_k t/t^2 + \ldots}_{=d}$ $\lrcorner$

<u>proof of Cor. 4.4.</u>: If $\mathcal{O}_P(F)$ is a DVR, let $m_P(F) = (x)$

$$\begin{array}{ccccc} m_P(F)^n = (x^n) & \longrightarrow & \mathcal{O}_P(F) & \twoheadrightarrow & k \\ \lambda \cdot x^n & \longmapsto & \lambda & \longmapsto & \lambda \bmod m_P(F) \end{array}$$

is surj. w/ kernel $m_P(F)^{n+1}$ $\forall n \geq 0$

$\overset{\text{Thm 4.1}}{\rightsquigarrow}$ $m_P(F) = \dim_k\left(m_P(F)^n / m_P(F)^{n+1}\right) = 1 \rightsquigarrow P$ is simple.

Conversely let P be a simple point, say $P = (0,0)$.

Let T be the unique tangent to F at P and $L \neq T$ as in the statement.

$\exists A \in GL_2(k)$ s.t. $A \cdot T = \{Y = 0\}$ and $A \cdot L = \{X = 0\}$.

Since $F \circ A \simeq F$ as varieties, we may assume $T = Y$ and $L = X$.

Then $F = Y +$ terms of higher degrees.

Now we need to show that $m_P(F) = (X, Y) \subset \mathcal{O}_P(F)$ is principal, generated by X:

Write $F = (\text{All monom. containing } Y) + \text{Rest}$

$= Y\left(1 + H(X,Y)\right) + X^2 \cdot G(X)$, w/ $H(0,0) = 0$

$\rightsquigarrow$ The image of $1 + H(X,Y)$ in $\mathcal{O}_P(F)$ is a unit.

$\rightsquigarrow$ The image of $1+H(X,Y)$ in $\mathcal{O}_P(F)$ is a unit.

$\rightsquigarrow$ $y = x^2 G(x) \cdot (1+H)^{-1} \in (X)$ in $\mathcal{O}_P(F)$.

$\rightsquigarrow$ $(X,Y) = (X)$ is principal, generated by the image of $L = X$. We have already seen that $\mathcal{O}_P(F)$ is Noeth. and local in Prop. 2.6.

$\rightsquigarrow \mathcal{O}_P(F)$ is a DVR $\square$

4.3 Intersection numbers:

Def: Let $P \in \mathbb{A}^2$, F, G two plane curves. We define the intersection number $I(P, F \cap G)$ of F and G at P by

$$I(P, F \cap G) := \dim_K \left(\mathcal{O}_P(\mathbb{A}^2)/(F,G)\right) \in \mathbb{N} \cup \{\infty\}.$$

The definition doesn't seem very intuitive. Let's compute it in the simplest case.

Def: Two plane curves F and G intersect transversally at a point $P \in F \cap G$ if P is a simple point for both F and G and the unique tangents of F and G at P are different.

Exple: $\cdot$ $I(P, F \cap G) = 1$ if F and G intersect trans. at P:
Since localization is exact we always have (assume F irred)

$$\mathcal{O}_P(\mathbb{A}^2)/_{(F,G)} \cong \left(K[X,Y]/_{(F,G)}\right)_{m_P} \cong \left(\Gamma(F)/_{G}\right)_{m_P} \cong \mathcal{O}_P(F)/_{(g)}$$

g is the image of G under $K[X,Y] \twoheadrightarrow \Gamma(F) \hookrightarrow \mathcal{O}_P(F)$.

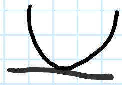
By Cor. 4.4, $\mathcal{O}_P(F)$ is a DVR. Since $G(P) = 0$, $g \in m_P \subset \mathcal{O}_P(F)$.

Again by Cor. 4.4, the tangent L of G at P is a unif. param.

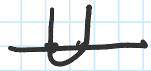
We have $G = L + G'$ w/ G' of deg $\geq 2 \Leftrightarrow G' \in m_P^2$

Thus in $\mathcal{O}_P(F)$, $g' \in (\ell)^2 \rightsquigarrow g = \ell(1 + g'/\ell) \rightsquigarrow (g) = (\ell)$

$\rightsquigarrow \mathcal{O}_P(F)/_{(g)} = \mathcal{O}_P(F)/_{(\ell)} \cong K \rightsquigarrow I(P, F \cap G) = 1$ as expected.

· $F=Y$, $G=X^2-Y$, $P=(0,0)$ 

Then $\mathcal{O}_P(\mathbb{A}^2)/(F,G) \cong k[X]/X^2 \rightsquigarrow I(P,F\cap G)=2$

This matches the ~~the~~ intuition, that if we move F or G a bit we get two intersection points .

<u>Thm 4.5</u>: The intersection number satisfies the following properties:

1) $I(P,F\cap G)=\infty \iff P$ lies on a common component of F and G
i.e. $\exists H$ s.t. $H|F$, $H|G$ and $H(P)=0$.

2) $I(P,F\cap G)=0 \iff P \notin F\cap G$

3) $I(P,F\cap G)$ depends only on the components of F and G that pass through P.

4) $I(P,F\cap G)=I(P,G\cap F)$

5) $I(P,F\cap G)\geq m_P(F)m_P(G)$ with equality if and only if F and G have no tangent lines in common at P.

6) If $F=\prod F_i^{r_i}$, $G=\prod_j G_j^{s_j}$, then $I(P,F\cap G)=\sum_{i,j} r_i s_j I(P,F_i\cap G_j)$

7) $I(P,F\cap G)=I(P,F\cap(G+F\cdot A))$ for any $A\in k[X,Y]$.

8) If P is a simple point of F (F irred.) then $I(P,F\cap G)=\mathrm{ord}_P^F(G)$

9) If F and G have no common components then

$$\sum_{P\in F\cap G} I(P,F\cap G)=\dim_k(k[X,Y]/(F,G))$$

pf: 1+9) If F, G have no common comp., then $\underset{\substack{\| \\ |F\cap G|}}{|V(F,G)|}<\infty$

by Lemma 1.9. Thus by Lemma 4.2.

$$k[X,Y]/(F,G) \simeq \prod_{P\in F\cap G} \mathcal{O}_P(\mathbb{A}^2)/(F,G)$$

Both sides are finite-dim k-vect. spaces by Cor 1.7.

Taking dim. on both sides gives 9) and i.p. $I(P,F\cap G)<\infty$.

If H is a $\overbrace{\text{common}}^{\text{irred.}}$ comp. of F, G containing P, we have

If H is a common $\overbrace{\text{comp.}}^{\text{irred.}}$ of F, G containing P, we have
$(F,G) \subset (H) \rightsquigarrow \mathcal{O}_P(\mathbb{A}^2)/_{(F,G)} \twoheadrightarrow \mathcal{O}_P(\mathbb{A}^2)/_H \cong \mathcal{O}_P(H) \supset \mathcal{O}(H)$
$\rightsquigarrow I(P, F \cap G) \geq \dim_K \Gamma(H) = \infty$ by Cor. 1.7. $\Rightarrow$ 1)

2) $p \in F \cap G \Leftrightarrow m_p \supset (F,G) \Leftrightarrow (F,G) \subsetneq \mathcal{O}_P(\mathbb{A}^2) \Leftrightarrow I(P, F\cap G) > 0$

3) If $F = F_1 \cdot F_2$ w/ $F_2(P) \neq 0$. Then F and F_1 differ by a unit in $\mathcal{O}_P(\mathbb{A}^2) \rightsquigarrow (F,G) = (F_1, G)$ $\Rightarrow$ 3).

4) $(F,G) = (G,F)$.

6) It's enough to show $\forall F, G, H \in K[X,Y]$

$$I(P, F \cap G \cdot H) = I(P, F\cap G) + I(P, F \cap H)$$

Assume F and $G \cdot H$ have no common component, otherwise both sides are ∞.
The statement follows from exactness of the following seq:
(Write $\mathcal{O} = \mathcal{O}_P(\mathbb{A}^2)$)

$$0 \to \mathcal{O}/_{(F,H)} \xrightarrow{\psi} \mathcal{O}/_{(F, G\cdot H)} \overset{\varphi}{\twoheadrightarrow} \mathcal{O}/_{(F,G)} \to 0$$

quotient morph.

$$\overline{z} \mapsto \overline{Gz}$$

Since loc. is exact, we can replace $\mathcal{O}$ by $K[X,Y]$.
φ surj ✓.
<u>ψ is inj:</u> $z \in K[X,Y]$ s.t. $G \cdot z \in (F, G\cdot H)$. Then

$$G \cdot z = A \cdot F + B \cdot GH \Leftrightarrow AF = G(z - BH)$$

Since F and G have no common comp. $F \mid z - BH \Leftrightarrow z - BH = F \cdot C$
$\rightsquigarrow z \in (F,H) \rightsquigarrow \overline{z} = 0$. ✓
<u>$\operatorname{Im}\psi = \ker\varphi$</u>: is similar. $\rightsquigarrow$ 6)

7) $(F,G) = (F, G + A\cdot F)$ in $\mathcal{O}_P(\mathbb{A}^2)$.

8) We have seen that

of we have seen that

$$\mathcal{O}_P(\mathbb{A}^2)/(F,G) \cong \mathcal{O}_P(F)/(g)$$

Taking dim. gives 8) $\qquad$ 1-4) and 6-9) $\square$

Before proveing 5) we look at the application of the thm.

Using this theorem one can compute $I(P,F\cap G)$ in an algorithmic way: First we check whether $P\in F\cap G$. If not we're done by 2). Otherwise, by a translation, we may assume $P=(0,0)$.

Let $r=\deg(F(X,0))$ $s=\deg(G(X,0))$ and w.l.o.g. $r\leq s$.

<u>1. case $r=0$:</u> $F=Y\cdot H$ for some H. and

$$I(P,F\cap G)\overset{6)}{=}I(P,Y\cap G)+I(P,H\cap G) \qquad G(0,0)=0$$

Write $G(X,0)=X^m(a_0+a_1X+\ldots+a_{s-m}X^{s-m})$, w/ $a_0\neq 0$ $\quad$ If $m=0 \to Y|G \Rightarrow I=\infty$

If $m>0$:

$$I(P,Y\cap G)\overset{7)}{=}I(P,Y\cap G(X,0))\overset{6)+2)}{=}I(P,Y\cap X^m)\overset{5)}{=}m$$ and we're left with computing $I(P,H\cap G)<I(P,F\cap G)$.

<u>2. case $r>0$:</u> We may assume $F(X,0),G(X,0)$ are monic. Let $H=G-X^{s-r}F$. Then $I(P,F\cap G)\overset{7)}{=}I(P,F\cap H)$ and $\deg(H(X,0))<s$. Repeating this we eventually get to case 1.

<u>Rmk:</u> In this algorithm we only used the properties 1)–7), not the definition. This shows that $I(P,F\cap G)$ is uniquely determined by 1)–7).

<u>Exple:</u> Let $F=(X^2+Y^2)^2+3X^2Y-Y^3$, $G=(X^2+Y^2)^3-4X^2Y^2$, $P=(0,0)$.

$$I(P,F\cap G)=I(P,F\cap(G-(X^2+Y^2)F))=I(P,F\cap (X^2+Y^2)Y(3X^2-Y^2)+4X^2Y^2)$$

$$=I(P,F\cap Y)+I(P,F\cap \underbrace{(X^2+Y^2)(3X^2-Y^2)+4X^2Y)}_{H})$$

$I(P,F(X,0)\cap Y)=4$ $\qquad$ ↳ eliminate terms without Y i.e. $3X^4$

$$I(P,F\cap H) = I(P,F\cap H-3F) = I(P,F\cap Y(\underbrace{5X^2-3Y^2+4Y^3+4X^2Y}_{J}))$$
$$= \underbrace{I(P,F\cap Y)}_{=4} + I(P,F\cap J)$$

But the lowest order term of F and J are $3X^2Y-Y^3$ and $5X^2-3Y^2$ and they have no common factor.

5) $\rightsquigarrow I(P,F\cap H)=6$ and in total $I(P,F\cap G)=14$.

We're still lacking the proof of part 5) of Thm 4.5:

$I(P,F\cap G) \geq m_P(F)m_P(G)$ with equality if and only if F and G have no tangent lines in common at P.

<u>pf:</u> Recall the definition $I(P,F\cap G) = \dim_k \mathcal{O}_P(\mathbb{A}^2)/(F,G)$

Write $m=m_P(F)$ and $n=m_P(G)$ and assume $P=(0,0)$, so that $m_P := I = (X,Y)$. Consider the diagram

$$\begin{tikzcd}
k[X,Y]/I^n \times k[X,Y]/I^m \arrow[r, "\psi"] & k[X,Y]/I^{m+n} \arrow[r, two heads, "\varphi"] & k[X,Y]/(I^{m+n},F,G) \arrow[r] \arrow[d, "\alpha", "\wr"'] & 0 \\
& \mathcal{O}_P(\mathbb{A}^2)/(F,G) \arrow[r, two heads, "\pi"] & \mathcal{O}_P(\mathbb{A}^2)/(I^{m+n},F,G) &
\end{tikzcd}$$

Here α, φ and π are the natural ring morph. and $\psi(\bar{A},\bar{B}) = \overline{A\cdot F + B\cdot G}$.

Since $V(I^{m+n},F,G)=\{P\}$, α is an isom. by Lemma 4.2.

Also the top-row is exact i.e. $\operatorname{Ker}\varphi = \operatorname{Im}\psi$:

$\bar{z}\in \operatorname{Ker}\varphi \iff z \in (I^{m+n},F,G) \iff z \equiv AF+BG \mod I^{m+n}$
$\Rightarrow \bar{z} = \psi(\bar{A},\bar{B})$.

$$I(P,F\cap G) = \dim_k \mathcal{O}_P(\mathbb{A}^2)/(F,G) \geq \dim_k \mathcal{O}_P(\mathbb{A}^2)/(I^{m+n},F,G)$$

$$\ldots (F,G) \geq \dim_k \mathcal{O}_P(\mathbb{A}^2)/(I^{m+n}, F, G)$$
$$= \dim_k k[X,Y]/(I^{m+n}, F, G)$$
$$\geq \dim_k k[X,Y]/I^{m+n} - \dim_k\left(k[X,Y]/I^{n} \times k[X,Y]/I^{m}\right)$$
$$= m \cdot n.$$

$\rightsquigarrow I(P, F \cap G) \geq m \cdot n$ w/ equality $\iff \pi$ is an iso and ψ is injective.

Thus 5) follows from the following two observations:

a) If F and G have no common tangents at P then
$$I^t \subset (F,G)\,\mathcal{O}_P(\mathbb{A}^2) \quad \text{for } t \geq m+n-1$$
In part. π is an iso in this case.

b) ψ is injective $\iff F$ and G have no common tangents.

<u>pf of a)</u> First we show that $I^t \subset (F,G)$ for large t:

Let $V(F,G) = \{P, Q_1, \ldots, Q_s\}$ and choose H s.t. $H(P) \neq 0$ and $H(Q_i) = 0$ $\forall i$

$\rightsquigarrow H \cdot X, H \cdot Y \in I(V(F,G)) \overset{NSS}{\rightsquigarrow} \exists N$ s.t. $(HX)^N, (HY)^N \in (F,G)$

But H^N is a unit in $\mathcal{O}_P(\mathbb{A}^2) \rightsquigarrow X^N, Y^N \in (F,G) \rightsquigarrow I^{2N} \subset (F,G)$.

Now let $L_1, \ldots, L_m$ be the tangents to F at P (with multiplicity).
$M_1, \ldots, M_n$ " G ".

Set $A_{ij} = L_1 \cdots L_i \cdot M_1 \cdots M_j \in I^{i+j}$ for $i,j \geq 0$

where we set $L_i = L_m$ for $i > m$ and $M_j = M_n$ for $j > n$.

Then $\{A_{ij} \mid i+j = t\}$ is a basis of I^{i+j}/I^{i+j+1} as k-v.sp (Ex. 10.1)

$\rightsquigarrow \{A_{ij} \mid i+j = t\}$ are generators for I^{i+j}.

Hence we need to show $A_{ij} \in (F,G)$ if $i+j \geq m+n-1$.

Hence we need to show $A_{ij} \in (F,G)$ if $i+j \geq m+n-1$.

W.l.o.g $i \geq m$. $\leadsto A_{ij} = A_{m0} \cdot B$ w/ $B \in I^{i+j-m}$

and we can write $F = A_{m0} + F'$ w/ $F' \in I^{m+1}$

$\leadsto A_{ij} - B \cdot F = -B \cdot F' \in I^{i+j+1}$.

We can repeat this w/ elements in I^{i+j+1}, $\leadsto I^{i+j+2}$ ....

and since we know that $I^t \subset (F,G)$ for large t

we see that $A_{ij} \in (F,G)$ for $i \geq m+n-1$ □

b) $\Leftarrow$: Let $A,B \in K[X,Y]$ w/ $\psi(\bar{A}, \bar{B}) = \overline{AF+BG} = 0$.

i.e. $AF+BG \in I^{m+n} \subset K[X,Y]$.

Write $A = A_r + .. + A_d$ and $B = B_s + ... + B_{d'}$

w/ A_i, B_j forms of deg i and j.

We need to show $r \geq n$ and $s \geq m$. Assume $r < n$, then

$I^{m+n} \ni AF+BG = A_r F_m + B_s \cdot G_n$ + higher order terms.

$\overset{r<n}{\leadsto} A_r F_m = -B_s \cdot G_n$. By assumpt. F_m an G_n have no common factor, and so $G_n | A_r$ and $F_m | B_s$ $\leadsto n \leq r$, $m \leq s$ ↯

$\Rightarrow$: Assume L is a common tangent. Then $F_m = L \cdot F'_{m-1}$

$G_n = L \cdot G'_{n-1}$

Then $\psi(\overset{\neq 0}{\overline{G'_{n-1}}}, \overset{\neq 0}{\overline{F'_{m-1}}}) = \overline{G'_{n-1}F - F'_{m-1}G} = 0$ and

ψ is not injective ✓ 5)

□